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TECHNICAL NOTES

FREE CONVECTION ABOUT A SPHERE AT SMALL GRASHOF NUMBER

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NOMENCLATURE

g', acceleration of gravity;
Gr, Grashof number, $g'\beta'(T'_w - T'_\infty)r'^3_w/v'^2$;
I_n, Gegenbaur functions of the first kind of order n;
h, $\bar{h}$, local and mean heat transfer coefficient;
K', thermal conductivity;
Nu, local Nusselt number based on r'_w;
$\overline{Nu}$, mean Nusselt number based on diameter of the sphere;
Pr, Prandtl number, v'/α';
P_n, Legendre polynomials of the first kind and of order n;
r', r, radial coordinate, $r = r'/r'_w$;
Ra, Rayleigh number, $GrPr$;
T', T, temperature, $T' = T'_\infty + (T'_w - T'_\infty)T$.

Greek symbols

α', thermal diffusivity;
β', volumetric coefficient of thermal expansion;
θ, colatitude or polar angle measured from upward vertical $\theta = 0$;
ρ', v', density and kinematic viscosity;
ψ', ψ, stream function, $\psi = \psi'/v'$;
ξ, $\ln(r'/r'_w)$;
δ, step size.

Subscripts and superscripts

$'$, dimensional quantity (unprimed quantities are dimensionless);
w, denotes value on the sphere;
∞, denotes value of the ambient fluid (at infinity).

1. INTRODUCTION

Natural convection from a sphere is governed by three dimensionless parameters: the Grashof number Gr, the Prandtl number Pr, and the space ratio r_∞ defined as the ratio of the dimensions of the enclosure or test section to the radius of the sphere. The problem has been extensively studied both theoretically and experimentally [1–7]. Recently Geoola and Cornish [8] presented the numerical solution of steady-state free convective heat transfer from a solid sphere for Grashof numbers in the range 0.05–50, Prandtl number equal to 0.72, and r_∞ equal to 25. They observed that the numerical scheme failed to find solution for Grashof numbers greater than 50.

In this paper, we numerically calculate the flow properties of the free convection problem about an isothermally heated sphere by the series truncation method when the Grashof number is of order unity. In fact, the series truncation method is used to solve the free convection between concentric spheres. A method similar to that used by Dennis and Singh [9] for the flow between two rotating spheres and Singh and Chen [10] for free convection problem is used in the present paper. The stream function and the temperature distribution are expanded as series of orthogonal Gegenbaur functions and Legendre polynomials, respectively, in terms of the angle θ of spherical polar coordinates (r', θ, ϕ). The finite set of equations which results from a certain level of truncation is solved numerically for $Pr = 0.7$ and Grashof numbers in the range 0.01–1.

2. ANALYSIS

Consider an otherwise undisturbed viscous fluid at constant temperature T'_∞ and in a uniform gravity field acting vertically downward. A sphere is introduced whose surface is maintained at a constant temperature $T'_w (> T'_\infty)$. The transformation $\xi = \ln(r'/r'_w)$ is introduced after which the Navier–Stokes equations for steady, axisymmetric motion can be obtained as [10],

$$D^2\psi = -e^{2\xi}\zeta, \tag{1}$$

$$D^2\zeta = -Gr\, e^{2\xi} \sin\theta \left(\sin\theta \frac{\partial T}{\partial \xi} + \cos\theta \frac{\partial T}{\partial \theta}\right) + \frac{e^{-\xi}}{\sin\theta}\left[\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\theta} + 2\zeta\left(\cot\theta\frac{\partial\psi}{\partial\xi} - \frac{\partial\psi}{\partial\theta}\right)\right)\right], \tag{2}$$

$$\nabla^2 T = \frac{Pr\, e^{-\xi}}{\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial\xi} - \frac{\partial\psi}{\partial\xi}\frac{\partial T}{\partial\theta}\right) \tag{3}$$

where

$$D^2 \equiv \frac{\partial^2}{\partial\xi^2} - \frac{\partial}{\partial\xi} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial\xi^2} + \frac{\partial}{\partial\xi} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

The boundary conditions are

$$\psi = \partial\psi/\partial\xi = 0, \quad T = 1, \quad \text{at} \quad r = 1 \quad \text{or} \quad \xi = 0, \tag{4}$$

$$e^{-2\xi}\psi \to 0, \quad e^{-2\xi}\partial\psi/\partial\xi \to 0, \quad T \to 0 \quad \text{as} \quad r \to r_\infty \quad \text{or} \quad \xi \to \xi_\infty. \tag{5}$$

Solution of equations (1)–(3) subject to conditions (4) and (5) depends on three parameters Gr, Pr and ξ_∞. We attempt to solve the above mentioned system for a fixed $Pr = 0.7$, small values of Gr and varying ξ_∞ from $\xi_\infty = \ln 2$ to $\ln 80$.

In order to apply the series truncation method, expansions for T, ψ and ζ are assumed as ($\mu = \cos\theta$)

$$T = \sum_{n=1}^{\infty} f_n(\xi)P_{n-1}(\mu), \quad (6)$$

$$\psi = \sum_{n=1}^{\infty} g_n(\xi)I_{n+1}(\mu), \quad (7)$$

$$\zeta = \sum_{n=1}^{\infty} h_n(\xi)I_{n+1}(\mu). \quad (8)$$

The Legendre polynomials P_n and Gegenbaur functions I_n are suitable for the operators ∇^2 and D^2 in equations (3) and (1) and (2), respectively (see refs. [9, 10] for definition and properties of I_n). After substitution of equations (6)–(8) into equations (1)–(3), and with the help of orthogonal and various integral properties of P_n and I_n functions, we obtain

$$g_n'' - g_n' - n(n+1)g_n = -e^{2\xi}h_n, \quad (9)$$

$$h_n'' - h_n' - n(n+1)h_n = S_n, \quad (10)$$

$$f_n'' - f_n' - n(n-1)f_n = R_n \quad (11)$$

where primes denote differentiation with respect to ξ. The quantities R_n and S_n can all be expressed in terms of f_n, g_n, h_n and the $3-j$ symbols [10]. The conditions for f_n, g_n and h_n can be obtained with the help of equations (4), (5) and (9).

$$f_1(0) = 1, \quad f_1(\xi_\infty) = 0, \quad f_n(0) = 0, \quad f_n(\xi_\infty) = 0 \quad \text{for} \quad n \geq 2, \quad (12)$$

$$g_n(0) = 0, \quad g_n(\xi_\infty) = 0, \quad g_n'(0) = 0, \quad g_n'(\xi_\infty) = 0 \quad \text{for all } n, \quad (13)$$

$$h_n(0) = -g_n''(0), \quad h_n(\xi_\infty) = -e^{-2\xi_\infty}g_n''(\xi_\infty) \quad \text{for all } n. \quad (14)$$

The three sets of second-order ordinary differential equations (9)–(11) are to be integrated subject to conditions (12)–(14). In numerical computation, the set of equations is truncated. A truncation of order m is defined by putting all functions in the expansion (6)–(8) with subscripts $n > m$ identically equal to zero. The resulting $3m$ equations are then solved with the associated boundary conditions. Since there are more conditions given by equations (13) than required to solve equation (9), a simple step-by-step procedure is constructed to integrate equation (9), where no iteration is required. The range $\xi = 0$ to $\xi = \xi_\infty$ is divided into 100 intervals of uniform grid size δ. The derivatives in equations (10) and (11) are approximated by means of standard three-point central difference formulas and the Gauss–Seidel iteration scheme is applied to solve for h_n and f_n at all grid points. Details of the integration method have been described by Singh and Chen [10].

The integration scheme is started, say for $Gr = 0.01$ and $Pr = 0.7$, by assuming an initial approximation for the first truncation $n_0 = 1$ given by the conduction solution

$$f_1^{(0)}(\xi) = 2\exp(-\xi)-1, \quad g_1^{(0)}(\xi) = g_1'^{(0)}(\xi) = h_1^{(0)}(\xi) = 0 \quad (15)$$

where $\xi_\infty = \ln 2$ or $r_\infty = 2$. The functions f_n, g_n, h_n for $n > 1$ are all taken to be zero. By employing the Gauss–Seidel iteration process, new approximations of $f_1(\xi)$ and $h_1(\xi)$ are obtained at all the grid points. Equation (14) determines the intermediate estimates of $h_1^{(1/2)}(0)$ and $h_1^{(1/2)}(\xi_\infty)$. First approximations to $h_1^{(1)}(0)$ and $h_1^{(1)}(\xi_\infty)$ are defined by ($m = 0$)

$$h_1^{(m+1)}(0) = sh_1^{(m+1/2)}(0) + (1-s)h_1^{(m)}(0), \quad (16)$$

$$h_1^{(m+1)}(\xi_\infty) = sh_1^{(m+1/2)}(\xi_\infty) + (1-s)h_1^{(m)}(\xi_\infty) \quad (17)$$

where s, known as a smoothing factor, is a number in the range $(0 < s < 1)$. Then the step-by-step integration scheme calculates $g_1^{(1)}(\xi)$ and $g'^{(1)}(\xi)$ at all grid points. This completes the first cycle of the iterative procedure. Using the most recent calculated values of the functions $f_1(\xi), \ldots h_1(\xi)$, etc. as the starting values, the sequence of iteration is continued until convergence is achieved. This is decided by the test (m denoting the number of iteration)

$$|h_1^{(m+1)}(\xi) - h_1^{(m)}(\xi)| < \varepsilon \quad (18)$$

where ε is a small preassigned tolerance, of the order of 10^{-5}.

After obtaining the converged solution for the first truncation $f_1(\xi)$, $g_1(\xi)$, $g_1'(\xi)$, and $h_1(\xi)$, the second and third truncations for $Gr = 0.01$, $Pr = 0.7$ and $\xi_\infty = \ln 2$ are iterated using the most recent values of the functions as input and repeating the above-mentioned procedure. Once the third truncation for a certain value of ξ_∞ has been solved, ξ_∞ is increased and the calculation is repeated. Numerical solutions are thus performed, each time increasing the value of ξ_∞ and using the most recent calculated values as input. When the calculated results near the sphere for two values of ξ_∞ (one larger than the other) differ by less than 1%, the larger value of $\xi_\infty(r_\infty = e^{\xi_\infty})$ fixes the outer sphere to be at a very large distance to have any effect on the inner sphere. According to this criterion, results for a single sphere immersed in an infinite medium are obtained.

The smoothing factor s, preassigned tolerance ε, the size of the grid δ and the number of truncation n_0 are all parameters of the solution. In Table 1, the values of the various parameters used in the calculations are listed.

Table 1.

Gr	ξ_∞	r_∞	s	ε	δ
0.01	4.044	60	0.15	10^{-5}	0.0205
0.05	4.044	60	0.15	10^{-5}	0.0205
0.10	4.044	60	0.1	10^{-5}	0.0205
0.20	4.044	60	0.1	10^{-5}	0.0205
	4.382	80			0.0219
0.50	4.044	60	0.1	10^{-4}	0.0205
1.0	4.044	60	0.1	10^{-4}	0.0205
	4.382	80	0.08		0.0219

4. DISCUSSION OF RESULTS

The local and average Nusselt numbers on the sphere are defined as

$$Nu = \frac{h' r_w'}{K'} = -\left[e^{2\xi}\frac{\partial T}{\partial \xi}\right]_{\xi=0}, \quad (19)$$

$$\overline{Nu} = \frac{\overline{h'}(2r_w')}{K'} = 2\int_0^{\pi}\left[e^{2\xi}\frac{\partial T}{\partial \xi}\right]_{\xi=0}\sin\theta\, d\theta. \quad (20)$$

The local Nusselt numbers have been calculated for $Gr = 0.2$ and 1, $Pr = 0.7$ and $r_\infty = 40$, 60 and 80. These values are given in Table 2. The local Nusselt numbers, isotherms and stream lines for $r_\infty = 60$ and 80 differ by less than 0.9%. The

Table 2. Comparison of local Nusselt number for $r_\infty = 40$, 60 and 80; $Gr = 0.2$, 1.0, $Pr = 0.7$.

	$Gr = 0.20$			$Gr = 1.0$		
θ	$r_\infty = 40$	$r_\infty = 60$	$r_\infty = 80$	$r_\infty = 40$	$r_\infty = 60$	$r_\infty = 80$
0	0.9743	0.9854	0.9870	0.9669	0.9770	0.9772
20	0.9826	0.9942	0.9962	0.9859	0.9958	0.9957
40	1.0060	1.0191	1.0218	1.0385	1.0480	1.0472
60	1.0399	1.0552	1.0591	1.1128	1.1222	1.1202
80	1.0784	1.0862	1.1015	1.1936	1.2036	1.200
100	1.1156	1.1359	1.1428	1.2671	1.2785	1.2738
120	1.1471	1.1695	1.1779	1.3243	1.3381	1.3321
140	1.1702	1.1943	1.2039	1.3625	1.3788	1.3718
160	1.1841	1.2091	1.2196	1.3833	1.4016	1.3939
180	1.1886	1.2141	1.2248	1.3898	1.4088	1.400
$\overline{Nu}$	2.1844	2.2222	2.2340	2.4280	2.4520	2.4440

Table 3. Local Nusselt number; $Pr = 0.7$.

	Gr					
θ	0.01	0.05	0.10	0.20	0.50	1.00
0	1.0094	0.9981	0.9932	0.9854	0.9825	0.9770
20	1.0103	1.0018	0.9991	0.9942	0.9963	0.9958
40	1.0129	1.0123	1.0158	1.0191	1.0348	1.0480
60	1.0167	1.0278	1.0403	1.0554	1.0899	1.1222
80	1.0214	1.0460	1.0686	1.0962	1.1514	1.2036
100	1.0263	1.0645	1.0965	1.1359	1.2093	1.2785
120	1.0308	1.0809	1.1207	1.1695	1.2567	1.3381
140	1.0345	1.0937	1.1391	1.1943	1.2903	1.3788
160	1.0368	1.1017	1.1505	1.2091	1.3098	1.4016
180	1.0376	1.1044	1.1543	1.2141	1.3161	1.4088
$\overline{Nu}$	2.0474	1.1080	2.1596	2.2222	2.3412	2.4520
$\overline{Nu}$[8]		2.09				2.39

fluctuations in the values of Nu for $Gr = 1$, persists for the third truncation. We believe that the fourth truncation should be calculated for $Gr = 1$. From this we conclude that for $Gr < 1$, results for the outer sphere being at a distance of $r_\infty = 60$ may be approximated for a single sphere in an infinite medium. As the Grashof number increases, the rate of heat transfer decreases at $\theta = 0$ and increases near $\theta = \pi$. The increase at $\theta = \pi$ is much more than the decrease at $\theta = 0$. The local Nusselt number for various values of Grashof number are given in Table 3. Table 3 also contains the values of mean Nusselt number calculated by the present method and those of Geoola and Cornish [8] for $Gr = 0.05$ and 1. Comparison is satisfactory. We infer from Fig. 1 that for small values of Gr in the range 0.01–0.1, isotherms are almost concentric spheres and the problem is conduction dominated. For $Gr = 1$ isotherms are nearer to one another at $\theta = \pi$ and tend to become apart near $\theta = 0$. This shows the tendency of plume formation near $\theta = 0$.

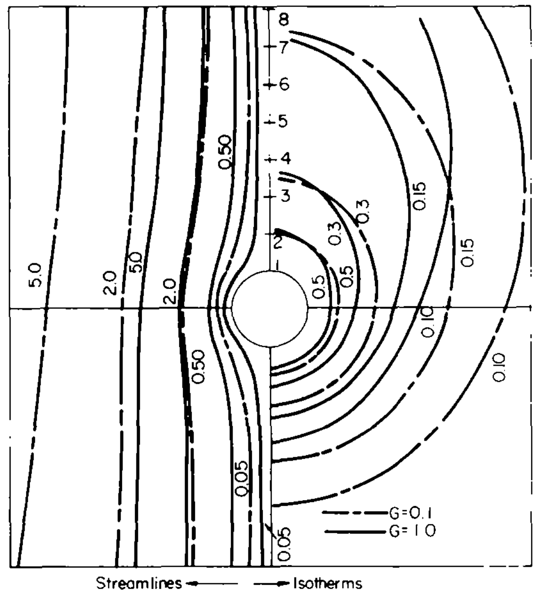

FIG. 1. Streamlines and isotherms for $Gr = 0.1$ and 1.0; $Pr = 0.7$.

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A SECOND LAW ANALYSIS OF THE CONCENTRIC TUBE HEAT EXCHANGER: OPTIMISATION OF WALL CONDUCTIVITY

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NOMENCLATURE

C,	minimum capacity rate $(\dot{m}c_p)_1$;
C_1, C_2,	capacity rate $(\dot{m}c_p)$ of smaller and larger flow rates, respectively;
D,	diameter of inner tube;
h,	heat transfer coefficient;
i,	overall inefficiency;
i_r,	inefficiency ignoring axial conductions but considering the thermal resistance of the partition wall;
I, I_1, I_2,	integrals defined in the text;